**Asymptotic Symmetry**

Let us consider the theory generated by the Lagrangian,

$$\mathscr{L} = \frac{1}{2}\left((\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2\right) - \frac{\lambda}{4!}\left(\phi_1^4 + \phi_2^4\right) - \frac{2\rho}{4!}\left(\phi_1^2 \phi_2^2\right).$$

From this Lagrangian we may compute the Feynman rules. We notice that while the ϕ_i^4 interaction has a symmetry of 4! to cancel the denominator, there is only a symmetry of 4 associated with the $\phi_1^2\phi_2^2$ vertex and therefore the vertex factor is $-i4 \cdot \frac{2\rho}{4!} = -i\frac{\rho}{3}$.

After we have renormalized with canonical renormalization conditions, the Feynman rules are: [1]
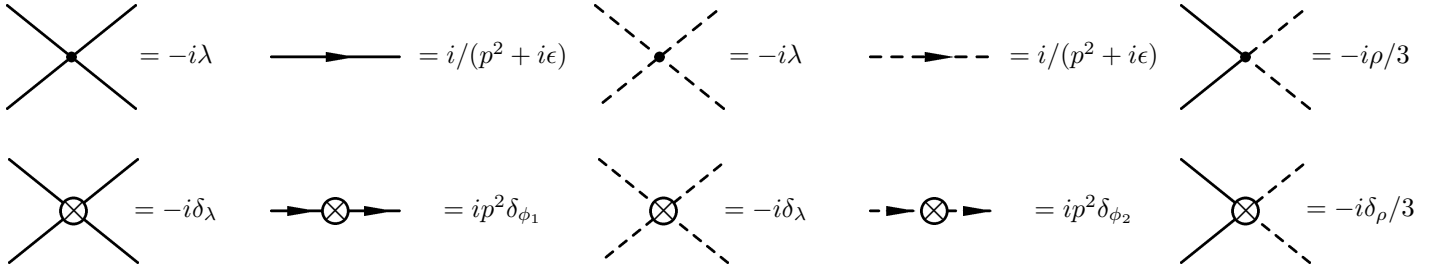
Let us now compute the β-functions for the couplings λ and ρ. To do this, we require the renormalization counter-terms δ_λ and δ_ρ.

To the one-loop order, we can find δ_λ by computing,

$$= -i\lambda + (-i\lambda)^2\left[V(t) + V(s) + V(u)\right] + (-i\frac{\rho}{3})^3\left[V(t) + V(s) + V(u)\right] - i\delta_\lambda,$$

$$= -i\lambda - \left(\lambda^2 + \frac{\rho^2}{9}\right)\left[V(t) + V(s) + V(u)\right] - i\delta_\lambda.$$

Now, we notice that the integral $V(k)$ is identical in all diagrams. In fact, every one-loop diagram we will concern ourselves with give the same loop integral $V(k)$. Let us compute the divergent piece of $V(k)$. Noticing the symmetry factor of $\frac{1}{2}$ and recalling our early results of dimensional regularization,

$$V(k) = \frac{1}{2}\int \frac{d^d k}{(4\pi)^d}\frac{i}{(k^2 + i\epsilon)}\frac{i}{((p-k)^2 + i\epsilon)},$$

$$= -\frac{1}{2}\int_0^1 dx \int \frac{d^d \ell}{(4\pi)^d}\frac{1}{(\ell^2 - \Delta)^2},$$

$$= -\frac{1}{2}\int_0^1 dx \frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Delta^{2-d/2}},$$

$$\underset{d\to 4}{\sim} -\frac{i}{32\pi^2}\frac{2}{\epsilon} \to -\frac{i}{32\pi^2}\log\frac{\Lambda^2}{M^2}.$$

Therefore, applying the canonical renormalization conditions, we see that

$$\delta_\lambda = \frac{3}{32\pi^2}\left[\lambda^2 + (\rho/3)^2\right]\log\frac{\Lambda^2}{M^2}.$$

Because there are no divergent self-energy diagrams in this theory to one-loop order[2], we have that the β-function for λ is given precisely by twice the coefficient of the log divergence in δ_λ.

$$\boxed{\therefore \beta_\lambda = \frac{3}{16\pi^2}\left[\lambda^2 + (\rho/3)^2\right].}$$

(1.b.1)

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[1]Notice that we have used ——▶—— to represent the field ϕ_1 and we have used ---▶--- to represent the field ϕ_2.

[2]It is clear that the ϕ^4 interaction does not offer any self-energy divergences to one-loop order. Furthermore, we see that the $\phi_1^2\phi_2^2$ interaction's contribution to self-energy also involves a loop independent of external momentum and therefore will not diverge.

Similar to our computation above, to find β_ρ we must compute the renormalization counter-term δ_ρ. To the one-loop order, we can find δ_ρ by computing,
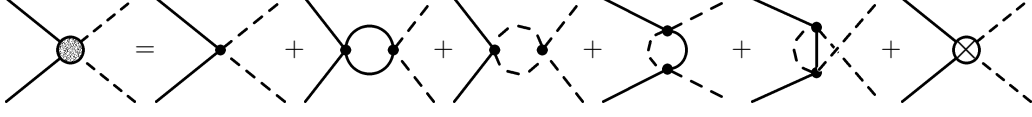
We notice that the symmetry factor of 2, included in our evaluation of the function $V(k)$, should not be included for the penultimate and antepenultimate diagrams because distinct fields run in the loop. Therefore, the loop integral for each of those two diagrams will contribute $2V(k)$ to to the total amplitude. Noting this subtlety, we find that

$$i\mathcal{M} = -i(\rho/3) + (-i\lambda)(-i\rho/3)\left[V(t) + V(t)\right] + (-i\rho/3)^2\left[2V(u) + 2V(s)\right] - i\delta_\rho/3.$$

Recall that we have already computed the divergence of the function $V(k)$ and noted that it was independent of k. Therefore,

$$i\delta_\rho/3 = (-i\lambda)(-i\rho/3)\left[V(t) + V(t)\right] + (-i\rho/3)^2\left[2V(u) + 2V(s)\right],$$

$$= -\lambda\rho/3\frac{-i}{16\pi^2}\log\frac{\Lambda^2}{M^2} - (\rho/3)^2\frac{-i}{8\pi^2}\log\frac{\Lambda^2}{M^2},$$

$$\therefore \delta_\rho = \frac{1}{16\pi^2}\left[\lambda\rho + 2\rho^2/3\right]\log\frac{\Lambda^2}{M^2}.$$

Because there are no divergent self-energy diagrams in this theory to one-loop order, we have that the β-function for ρ is given precisely by twice the coefficient of the log divergence in δ_ρ.

$$\boxed{\therefore \beta_\rho = \frac{1}{8\pi^2}\left[\lambda\rho + 2\rho^2/3\right].} \tag{1.b.2}$$

Let us now consider the β-function associated with the ration λ/ρ. Using the chain rule for differentiation and the definition of the general β-function, we see that

$$\beta_{\lambda/\rho} = \frac{1}{\rho^2}\left[\beta_\lambda\rho - \beta_\rho\lambda\right] = \frac{1}{\rho^2}\left[\frac{3\lambda^2\rho}{16\pi^2} + \frac{\rho^3}{48\pi^2} - \frac{\lambda^2\rho}{8\pi^2} - \frac{\rho^2\lambda}{12\pi^2}\right],$$

$$= \frac{(\lambda/\rho)^2\rho}{16\pi^2} + \frac{\rho}{48\pi^2} - \frac{(\lambda/\rho)}{12\pi^2},$$

$$= \frac{\rho}{48\pi^2}\left[3(\lambda/\rho)^2 - 4(\lambda/\rho) + 1\right],$$

$$\boxed{\therefore \beta_{\lambda/\rho} = \frac{\rho}{48\pi^2}\left(3\lambda/\rho - 1\right)\left(\lambda/\rho - 1\right).} \tag{1.c.1}$$

We see immediately that the two roots of $\beta_{\lambda/r}$ occur when $\lambda/\rho = 1, \frac{1}{3}$ and because the second derivative of $\beta_{\lambda/r}$ is $6 > 0$, we know that $\beta_{\lambda/\rho} < 0$ for $\lambda/\rho \in (\frac{1}{3}, 1)$ and $\beta_{\lambda/r} > 0$ for $\lambda/\rho > 1$. Therefore, for all $\lambda/\rho > \frac{1}{3}$, λ/ρ will flow to $\lambda/\rho = 1$. See Figure 1 below.

Therefore at large distances the couplings will flow to $\lambda = \rho$. This introduces a continuous $O(2)$ symmetry into the theory. To see this, let us define $\varphi \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. In this notation, the Lagrangian simply reads

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\lambda}{4!}\varphi^4.} \tag{1.e.1}$$

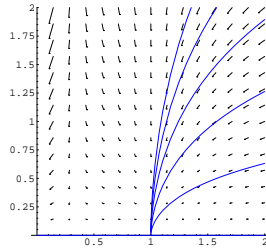This Lagrangian is clearly invariant to $O(2)$ transformations which correspond to changing the phase of φ.

FIGURE 1. Renormalization Group Flow as a function of scale. Arrows show $p \to 0$ flow.

Asymptotic Freedom

Let us consider a theory with a coupling constant g such that

$$\beta(g) = -\frac{\beta_1 g^3}{16\pi^2} \qquad \text{and} \qquad \gamma(g) = \frac{\gamma_1 g^3}{16\pi^2},$$

for some positive constants β_1, γ_1.

The renormalized correlation functions satisfy the Callan-Symanzik equations which, for the amputated correlators, take the form

$$\left[M\frac{d}{dM} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) \right] \Gamma_R^{(n)}(p_i/m, g) = 0.$$

If we take all the momenta to be equal for simplicity, then the solutions to the Callan-Symanzik equations take the form

$$\Gamma_R^{(n)}(p/M, g) = \Gamma^{(n)}(\overline{g}(p/M)) \exp\left(-4 \int_M^p d\log(p'/M)\gamma(\overline{g}(p'; g)) \right).$$

Let us compute the running coupling $\overline{g}(p/M)$. By the Callan-Symanzik equations, we see that

$$\frac{d\overline{g}}{d\log(p/M)} = -\frac{\beta_1 \overline{g}^3}{16\pi^2} \implies \int_g^{\overline{g}} \frac{d\overline{g}}{\overline{g}^3} = -\frac{\beta_1}{16\pi^2} \int d\log(p/M),$$

$$\implies -\frac{1}{2}\left(\frac{1}{\overline{g}^2} - \frac{1}{g^2} \right) = -\frac{\beta_1}{16\pi^2}\log(p/M),$$

$$\boxed{\therefore \overline{g}^2 = \frac{g^2}{1 + g^2 \frac{\beta_1}{8\pi^2}\log(p/M)}.} \tag{2.b.1}$$

Therefore, we see immediately that when $p/M \to \infty$, 1 becomes insignificant in the denominator of $\overline{g}^2$ and so $\overline{g}$ becomes independent of g. We see that

$$\boxed{\therefore \overline{g}^2 \underset{p\to\infty}{\approx} \frac{8\pi^2}{\beta_1 \log(p/M)}.} \tag{2.b.2}$$

Furthermore, we notice that this approximation can be trusted because nonperturbative effects become weaker at higher energy scales in an asymptotically free theory.

Let us now compute the dependence of the four-point vertex on momentum as $p/M \to \infty$. We assume that, to the lowest order, $\Gamma_R^{(4)} = g^2$. We cited the general solution to the (amputated) Callan-Symanzik equation above. Let us attempt to compute the integral in the exponent which multiplies $\Gamma^{(4)}(\overline{g})$. Using $\overline{g}$ from our work above, we see that

$$\int_M^p d\log(p'/M)\gamma(\overline{g}(p'; g)) = \int_M^p d\log(p'/M)\frac{\gamma_1}{16\pi^2}\frac{g^3}{(1 + g^2 \frac{\beta_1}{8\pi^2}\log(p'/M))^{3/2}},$$

$$= \frac{8\pi^2}{\beta_1 g^2}\frac{\gamma_1}{16\pi^2}\frac{-2g^3}{(1 + g^2 \frac{\beta_1}{8\pi^2}\log(p'/M))^{1/2}}\Bigg|_M^p,$$

$$= -\frac{\gamma_1 g}{\beta_1}\left[\frac{1}{(1 + g^2 \frac{\beta_1}{8\pi^2}\log(p/M))^{1/2}} - 1 \right],$$

$$\underset{p\to\infty}{\approx} \frac{\gamma_1 g}{\beta_1}.$$

Unfortunately, this result cannot be trusted in general. This is because a very large portion of this integral came from the lower bound $p' = M$ as $p \to \infty$. The energy scale M is usually chosen to represent the beginning of the non-perturbative regime in an asymptotically free field theory so our one-loop estimate of the functions $\beta(g), \gamma(g)$ cannot be trusted near $p = M$.

However, the calculation has taught us an important lesson. Although the precise value of the integral is largely uncalculable, the form of the solution is predicted. In particular, our evaluation of the integral showed us that whatever the result will be, it will be a constant, independent of p at large momenta. Therefore, using our work from above, the general four-point function will be of the form $\Gamma^{(4)}e^{\text{constant}} \propto \overline{g}^2$. Because we know the behavior of $\overline{g}^2$ as $p/M \to \infty$, we conclude that

$$\boxed{\therefore \Gamma^{(4)} \underset{p\to\infty}{\sim} \frac{8\pi^2}{\beta_1 \log(p/M)}.} \tag{2.c.1}$$